Discrete Homotopy and Homology Groups

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BIRS – Algebraic Combinatorixx 2
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Overview

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- Unexpected Application of Discrete Homotopy Theory

$$A_1^r(Cay(G/N)) \cong N$$

detects normal subgroups

Discrete Homotopy Theory for Graphs

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   $f(\vec{i}) = v_0$ almost everywhere
3. f, g are *discrete homotopic* if there exist $h \in \mathcal{A}_{n+1}(\Gamma, v_0)$ and $k, \ell \in \mathbb{N}$ such that for all $\vec{i} \in \mathbb{Z}^n$,
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4. $A_n(\Gamma, v_0)$ - set of equivalence classes of maps in $\mathcal{A}_n(\Gamma, v_0)$
   Note: translation preserves discrete homotopy

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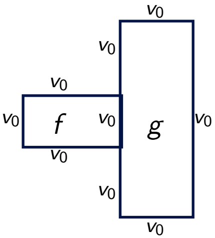

$$[f\,g] = [f]\,[g]$$

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f :

1	I	B	M	O	C
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f^{-1} :

1	C	O	M	B	I
0	N	A	T	O	R
-1	I	X	X	I	S
-2	S	U	P	E	R
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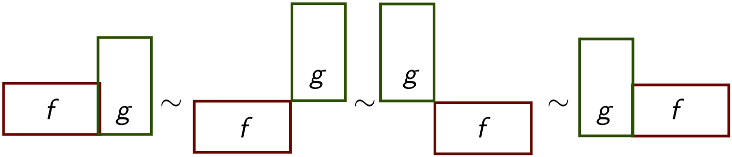

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$$A_1\left(v_0 \frac{\quad\quad\quad}{} v_1, v_0 \right) = 1$$

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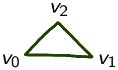
$A_1\Big($  $, v_0\Big) = 1$

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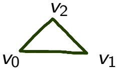
$A_1\Big($ 

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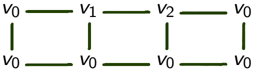

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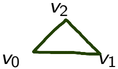
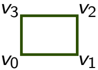
$A_1\Big($  $, v_0\Big) = 1$

$A_1\Big($  $, v_0\Big) = 1$

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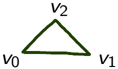
$A_1\Big($ 

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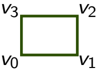
$A_1\Big($ 

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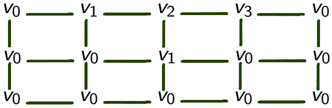

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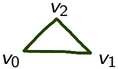
$A_1\big($ 

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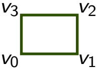
$A_1\big($ 

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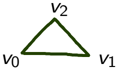
$A_1\big($ 

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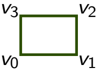
$A_1\big($ 

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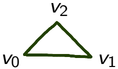
$A_1\Big($ 

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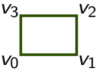
$A_1\Big($ 

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(2-dim cell complex: attach 2-cells to $\triangle$, $\square$ of Γ)

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- $A_n^q(\Delta, \sigma_0) \cong A_n(\Gamma_\Delta^q, \sigma_0)$

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- $A_n^r(X, x_0)$ r-Lipschitz maps $f : \mathbb{Z}^n \to X$ (stabilizing in all directions)

$$f : X \to Y \text{ is } r\text{-Lipschitz} \iff d(f(x_1), f(x_2)) \leq r\, d(x_1, x_2)$$

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4. Associated discrete homology theory. . . ?

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$$C_n(\Gamma) := \mathcal{L}_n(\Gamma)/D_n(\Gamma)$$

elements of C_n correspond to n-chains

Discrete Homology Theory for Graphs

Necessities

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- $\partial_n \circ \partial_{n+1} = 0$

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The *discrete homology groups* of Γ:

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The *relative homology* groups of (Γ, Γ'):

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   $$\pi_1(M) \cong A_1(\Gamma_R)$$
   - For each abelian group G and $\bar{n} \in \mathbb{N}$, there is a finite connected simple graph Γ such that
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   - There is a graph S^n such that

     $$DH_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Unexpected Application of Discrete Homotopy Theory

$S :=$ finite set
$G := \langle S \rangle$: finitely generated group
$Cay(G, S)$: graph with

- Vertex set: G
- Edge set: $\{(g, gs) : g \in G, s \in S\}$
- Label set: S

Note: a path from e to g is a word in S equal to g. Words along loops are relators in G (i.e: equal to e.)

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Theorem
If F_S is the free group on S and N is a normal subgroup of F_S, then

$$\pi_1(Cay(F_S/N, \overline{S}), e) \cong N$$

The fundamental group of the Cayley graph detects normal subgroups.

In general (when G is not free),

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Theorem (Delabie-Khukhro 2017)

$$A_{1,r}(Cay(G/N, \overline{S}), e) \cong N$$

for any constant r such that $2k \leq 4r < n$, where

$$k = \max\{|g|_{F_S} : g \in R\} \quad \textit{and} \quad n = \inf\{|g|_G : g \in N \setminus \{e\}\}.$$

The discrete fundamental group of the Cayley graph detects normal subgroups.

Thank-you!

Complex $K(\pi, 1)$ Spaces

$\mathcal{A}^{\mathbb{C}}_{n,2}$ braid arrangement:
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$\pi_1(M(W_{n,3})) \cong \mathrm{Ker}(\phi')$
where $W_{n,3}$ is a 3-parabolic
subgroup of type W
(B-Severs-White 2009)

Unexpected Application of Discrete Homotopy Theory

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Note: We have replaced a group (π_1) defined in terms of the topology of a space with a group (A_1) defined in terms of the combinatorial structure of the space.

What is Next?

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Homologies of path complexes and digraphs, by A. Grigoryan, Y. Lin, Y. Muranov, S.-T. Yau

A path complex P on a finite set V is a collection of paths (=sequences of points) on V such that if a path *v* belongs to P then a truncated path that is obtained from *v* by removing either the first or the last point, is also in P. Any digraph naturally gives rise to a path complex where allowed paths go along the arrows of the digraph.
A path complex P gives rise to a chain complex with an appropriate boundary operator δ that leads to the notion of path homology groups of P.

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Conjecture: Path homology and discrete homology groups are isomorphic for undirected graphs.

Note: Path complexes can be regarded as generalization of the notion of simplicial complexes. Any simplicial complex S determines naturally a path complex by associating with any simplex from S the sequence of its vertices.